Commonly used approaches to Learning, Teaching and Assessment in Mathematics.

The Mathematics Tuning group undertook an audit of, *inter alia*, modes of learning, teaching and assessment in the participating universities. The responses verify that teaching and learning takes place in combinations of the following:

Lectures.

- These are seen as a very time-efficient way for students to learn part of the large material involved in the corpus of mathematics. Self-study is quite difficult in mathematics, before an average student has acquired a sufficient basis of mathematical knowledge and skills, and much valuable time could be wasted if a student were to try to assimilate this material independently from the literature. In some cases, students acquire prepared lecture notes or have a set textbook; in other cases the taking of notes is seen as part of the learning process. The ideal approach here may be subject dependent. Nonetheless, lectures continue to play a dominant role in mathematics teaching, not merely in terms of imparting course content, but also with respect to the development of the key competences, and this was reflected in the responses of the group members.

- **Exercise sessions**. These are organised most often in tandem with lectures. They occur as groups with supervision, or individually as homework with subsequent supervision of the results. The aim of the exercises is two-fold: understanding of the theoretical material through examples and applications to problems. These sessions are essential in mathematics, where understanding is acquired by practice, not memorisation.

- **Homework.** While demanding on the time of the lecturer and/or teaching assistant, homework is clearly one of the most effective ways in which students can be encouraged to explore the limits of their capabilities. Group members have recorded its use in remedying deficiencies in students' knowledge of "elementary mathematics", which ideally should have been already acquired at secondary school. In this case it functions as a diagnostic test of the students' (developing) skills, which are developed in an additional voluntary course concerning these basic skills. Homework, of course, allows feedback to the students, which gives them a clearer picture of their performance; however, while homework is often assigned, it is less often graded, except where classes are small.

- **Computer laboratories.** These are perhaps the most significant change in the teaching of mathematics in recent years, introducing an experimental aspect to the subject. They feature not only in computer science related and computational courses, but also in statistics, financial mathematics, dynamical systems etc. Laboratory sessions are also seen as by far the best environment in which to develop the c"ability to use computational tools as an aid to mathematical processes and for acquiring further information, the importance of which can only increase as computer aided analysis becomes more common.

- **Projects**. These are done individually or in small groups, and typically involve putting together material from different sub-fields to solve more complicated problems. Small group projects can help to develop the ability to do teamwork (identified as an important transferable skill). The projects may involve significant computational elements, as in the case of the computational competences referred to above. Throughout, the emphasis should be on understanding the mathematics and its interpretation; thus on learning progressively to pass from a problem to its mathematical model, to the solution of the mathematical problem and finally to the interpretation of the solution in terms of the original problem. Projects, particularly significant final year projects where they exist, also afford the opportunity to develop students' verbal and written communications skills.

- **Search of bibliography**. Both in libraries and on the internet, familiarity with efficient ways to obtain relevant information must be acquired, in particular at second cycle level.

- **Dissertation**. In a second cycle or Masters programme, a substantial individual piece of work should be accomplished in the last year, as a final step towards independent practice of mathematics. It could take different forms depending on the sub-field, but would be characterised by its level and workload.

- **Mathematics Learning Centres.** As a mechanism to deal with some students' difficulties with the transition from secondary school to university level mathematics, many universities now have Mathematics Learning Centres, which typically provide flexible, student-centred approaches to learning for these students. Diagnostic tests are also used to identify gaps in their knowledge of elementary mathematics.

Feedback and assessment.

Assessment is mostly by written or oral end of semester examination, often supplemented by midterm examinations, homework exercises, and where relevant project assignments and programming assignments. If end of semester examinations are the sole assessment there is of course less feedback available to the students. On the other hand, lack of funding often limits the availability of continuous assessment. It has been noted that shortcomings in students' understanding of what is required of them often only becomes apparent at the time of assessment.

Final year projects and second cycle dissertations have feedback built in as part of the supervision process. Some students perform better in this situation than in the traditional examination format.

Conclusion.

Different modes of teaching and learning have a part to play in a mathematics programme, with some more appropriate to particular sub-fields and particular competences. In many universities, limitation of study time and lack of funding have pushed the balance towards time and cost efficient methods, mostly lectures and tutorials, except perhaps for a dissertation in the final year. The implementation of the Bologna declaration should be the opportunity to introduce more student centred teaching modes, to supplement the traditional ones.

Methodology of identifying competences

As part of the first phase of Tuning, academics were asked to rank 15 subject specific competences in terms of their importance for first and second cycles. All of the competences were considered important at either one or both cycles; the responses influenced the selection of the three key programme outcomes for Mathematics listed below and were also used to allot the competences to the two cycles for consideration. Nonetheless, most of the competences are developed to some extent at each cycle.

Tables of educational activities for subject specific competences at each cycle.

First cycle

1. Profound knowledge of “elementary” mathematics (such as may be covered in secondary education).
2. Ability to construct and develop logical mathematical arguments with clear identification of assumptions and conclusions.
3. Capacity for quantitative thinking.
4. Ability to extract qualitative information from quantitative data.
5. Ability to formulate problems mathematically and in symbolic form so as to facilitate their analysis and solution.
6. Ability to design experimental and observational studies and analyse data resulting from them.
7. Ability to use computational tools as an aid to mathematical processes and for acquiring further information.
8. Knowledge of specific programming languages or software.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Lectures		**X**	**X**	**X**	**X**	**X**	**X**	**X**
Exercise sessions	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
Homework	**X**	**X**	**X**	**X**	**X**	**X**		**X**
Computer Labs			**X**	**X**	**X**	**X**	**X**	**X**
Individual projects			**X**	**X**	**X**	**X**	**X**	**X**
Group projects							**X**	

Special assistance	**X**		**X**					

Second cycle

1. Facility with abstraction including the logical development of formal theories and the relationships between them.
2. Ability to model mathematically a situation from the real world and to transfer mathematical expertise to non-mathematical contexts.
3. Readiness to address new problems from new areas.
4. Ability to comprehend problems and abstract their essentials.
5. Ability to formulate complex problems of optimisation and decision making and to interpret the solutions in the original contexts of the problems.
6. Ability to present mathematical arguments and the conclusions from them with clarity and accuracy and in forms that are suitable for the audiences being addressed, both orally and in writing.
7. Knowledge of the teaching and learning processes of mathematics.

	1	**2**	**3**	**4**	**5**	**6**	**7**
Lectures	**X**	**X**	**X**	**X**	**X**		**X**
Exercise sessions	**X**	**X**	**X**	**X**	**X**		**X**
Homework			**X**		**X**	**X**	**X**
Computer Labs			**X**				**X**
Individual projects		**X**	**X**		**X**	**X**	**X**
Group projects		**X**	**X**		**X**		**X**
Dissertation			**X**	**X**		**X**	**X**

Process of competence development

It was noted previously[1] that the three key skills that any mathematics graduate should acquire are:

- the ability to conceive a proof,
- the ability to model a situation,
- the ability to solve problems.

[1] Towards a common framework for Mathematics degrees in Europe, Report of Tuning I, Julia Gonzalez and Robert Wagenaar (eds), 2002.

These are the key programme learning outcomes, reflected at different levels in the subject specific competences listed. Again it should be noted also that these competences are developed progressively throughout a programme: a mathematics programme does not contain any units specifically on proof construction, for example; rather, it is through practice in all course units that these skills are developed.

Commentary on competences

Profound knowledge of “elementary” mathematics

Students tend not to be aware of the importance of a profound knowledge of “elementary” mathematics until initial assessments, which clarify to them the necessity for rigour and completeness of solutions. Mathematics from secondary school is typically reviewed in early semesters, but without adequate time to compensate for any gaps in their knowledge. Diagnostic tests, feedback on homework and/or assessments and additional tutorial assistance are all helpful.

Ability to construct and develop logical mathematical arguments with clear identification of assumptions and conclusions.

This is typically developed by asking students for proofs and correcting them. When shown a corrected proof quite often they do not perceive the difference and simply consider the teacher too formal. It is not at all uncommon that students leave an exam thinking they have done well, but fail, because what they write is not at all perceived as a “logical argument” by the graders. Whether this capacity for strict logical thinking is innate (or needs to be learned very young), or can be taught at age 18-20 is a matter of debate.

Capacity for quantitative thinking.

This competence is so fundamental that it pervades all learning activities. There is some need to do computations without the aid of calculators, and most importantly to be aware of whether an answers is reasonable or not.

Ability to extract qualitative information from quantitative data.

Statistics

Ability to formulate problems mathematically and in symbolic form so as to facilitate their analysis and solution.

This competence essentially involves the ability to express a simple problem in the form of an equation, to express a statement written in common language in symbolic/mathematical form and vice versa, and to be critical about the solution, to know when a solution is sensible. This can be developed through feedback on exercises, and through problem solving and project work, where the application can illustrate the reasonableness of the solution.

Ability to design experimental and observational studies and analyse data resulting from them.

One interpretation of this competence is that first cycle students should be able to design functioning code segments in a high level language (e.g., Maple or Matlab), correct input errors (i.e., understand the mathematics of the syntax), and then interpret the data (for example a phase plane portrait). In general, as computer aided analysis becomes more and more common, ability to appropriately design experiments will become a skill of increasing importance. Lab sessions are the best environment in which to develop such skills.

Ability to use computational tools as an aid to mathematical processes and for acquiring further information.

This typically involves acquiring familiarity with a range of tools in a mathematical (e.g., MATLAB) , or specifically statistical (e.g., R, S Plus, GLIM) or more general (e.g., Java) context and learning how to use them in modelling and problem solving. Lab sessions and projects are the usual means of developing these skills

Knowledge of specific programming languages or software.

This concerns knowledge of the tools referred to the last competence, but also mathematical typesetting such as LaTeX, which can be used in project reports.

Facility with abstraction including the logical development of formal theories and the relationships between them.

This includes the following "abilities":

- understanding what mathematical objects are,
- manipulating them under formal rules,
- distinguishing between correct and incorrect operations,
- understanding the role of axioms, definitions and theorems.

Students are introduced to a variety of mathematical theories. They explore the limits of the theories under study, and they learn how some aspects of reality can be transformed into a formal theory after excluding what is considered accidental for the particular problem. They study and understand some theorems, perform some manipulations under some rules and check their work against the correct versions, which are supplied.

Ability to model mathematically a situation from the real world and to transfer mathematical expertise to non-mathematical contexts.

This enables students to enjoy genuine applied mathematics. Students learn to write the equations for a real life problem, for which the model has been (almost completely) suggested in native language. They are not taught to propose their own model; rather, they study examples, they clearly describe the limits and defaults of the proposed model and understand why this should not be a reason to reject the model. A good understanding of the modelled science seems essential.

Readiness to address new problems from new areas.

This is understood as the ability to identify mathematical concepts and abstractions in unknown domains. This may involve lots of case studies, a dissertation and could possibly be industrially linked. Individual or group project work is appropriate.

Ability to comprehend problems and abstract their essentials.

Course units in mathematical modelling can help to develop this competence. At second cycle a dissertation affords further opportunity to develop it in some depth. Assessment of the thesis will involve a judgement of this capacity.

Ability to formulate complex problems of optimisation and decision making and to interpret the solutions in the original contexts of the problems.

This is a particular example of problem solving. Through the second cycle, an increasing part of their coursework becomes project and problem oriented, as this is seen as the best way to develop complex skills. There is more learning than teaching: the instruction becomes one of advising (rather than dictating), and commenting (rather than grading). Students learn to interpret, and to discuss, the results of their work.

Ability to present mathematical arguments and the conclusions from them with clarity and accuracy and in forms that are suitable for the audiences being addressed, both orally and in writing.

In the second cycle, they should be able to

- recognise whether there are assumptions or hidden hypotheses in their arguments that have to be dealt with if they need to be removed;
- present an argument to their peers;
- give convincing arguments to instructors.

Knowledge of the teaching and learning processes of mathematics.

At second cycle, students will have been exposed to the varied teaching and learning processes as learners; acting as tutors for some first cycle courses, often to engineering, science or business classes, makes them think about good teaching methods.